Calculus 1
Midterm Exam – Solutions
September 30, 2024 (18:30 – 20:30)

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1) Prove using the (ε, δ)-definition that $\lim_{x \to 0} \sqrt{1 - x^2} = 1$.

Solution. Let $\varepsilon > 0$ be arbitrary. We consider that cases $\varepsilon > 1$ and $\varepsilon \le 1$ separately.

Start by noting that the range of the real function $f(x) = \sqrt{1 - x^2}$ is the closed interval $[0, 1]$, that is

$$0 \le \sqrt{1 - x^2} \le 1.$$

Therefore

$$-1 \le \sqrt{1 - x^2} - 1 \le 0$$

so by taking the modulus, we get

$$0 \le |\sqrt{1 - x^2} - 1| \le 1$$

This means that if $\varepsilon > 1$ then $|\sqrt{1 - x^2} - 1| \le 1 < \varepsilon$ holds for all x where $f(x)$ is defined, namely for all $|x| \le 1$. Thus any $\delta \le 1$ is a good choice, since for any $\delta \le 1$ having $0 < |x - 0| < \delta$ guarantees $|x| \le 1$ and therefore implies $|\sqrt{1 - x^2} - 1| < \varepsilon$.

If $\varepsilon \le 1$, then

$$|\sqrt{1 - x^2} - 1| < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt{2\varepsilon - \varepsilon^2}$$

as seen from the following equivalent inequalities:

$$
\begin{aligned}
|\sqrt{1 - x^2} - 1| &< \varepsilon && \text{(by definition)} \\
-\varepsilon < \sqrt{1 - x^2} - 1 &< \varepsilon && \text{(add } 1\text{)} \\
1 - \varepsilon < \sqrt{1 - x^2} &< 1 + \varepsilon && \text{(square each expression)} \\
(1 - \varepsilon)^2 < 1 - x^2 &< (1 + \varepsilon)^2 && \text{(subtract } 1\text{)} \\
(1 - \varepsilon)^2 - 1 < -x^2 &< (1 + \varepsilon)^2 - 1 && \text{(multiply by } -1\text{)} \\
1 - (1 + \varepsilon)^2 < x^2 &< 1 - (1 - \varepsilon)^2 && \text{(expand the brackets)} \\
-2\varepsilon - \varepsilon^2 < x^2 &< 2\varepsilon - \varepsilon^2 && \text{(use that } \varepsilon > 0 \text{ and } x^2 \ge 0 \text{ for real } x\text{)} \\
x^2 &< 2\varepsilon - \varepsilon^2 && \text{(take square roots; it's OK if } \varepsilon < 2\text{)} \\
|x| &< \sqrt{2\varepsilon - \varepsilon^2}
\end{aligned}
$$

This means that any $\delta \le \sqrt{2\varepsilon - \varepsilon^2}$ is a good choice as having $0 < |x - 0| < \delta$ implies $|x| < \sqrt{2\varepsilon - \varepsilon^2}$ which, as we saw, is equivalent to $|\sqrt{1 - x^2} - 1| < \varepsilon$.

To summarize, we showed that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|\sqrt{1 - x^2} - 1| < \varepsilon$. Therefore by the (ε, δ)-definition of limits we have $\lim_{x \to 0} \sqrt{1 - x^2} = 1$.

2) Apply l'Hospital's rule to evaluate the following limit: $\lim_{x \to \infty} \frac{x^{\ln x}}{(\ln x)^x}$. Indicate which rules of differentiation are being used in each step.

Solution. This limit is an indeterminate form of type "∞/∞" thus we may apply l'Hospital's Rule directly. To find the derivatives of the numerator and denominator, we use logarithmic differentiation. For the numerator, we find

$$(x^{\ln x})' = (x^{\ln x})\big(\ln(x^{\ln x})\big)' = (x^{\ln x})([\ln x][\ln x])' = (x^{\ln x})\left(2\frac{\ln x}{x}\right).$$

As for the denominator, we get

$$\big((\ln x)^x\big)' = \big[(\ln x)^x\big]\big[\ln\big((\ln x)^x\big)\big]' = \big[(\ln x)^x\big]\big[x\ln(\ln x)\big]' = \big[(\ln x)^x\big]\left(\ln(\ln x) + \frac{1}{\ln x}\right).$$

Above we used the Law of Logarithms that says $\ln(a^k) = k\ln a$ as well as the Product Rule, the Chain Rule and the derivatives $(x)' = 1$, $(\ln x)' = \frac{1}{x}$. L'Hospital's Rule yields

$$\lim_{x\to\infty}\frac{x^{\ln x}}{(\ln x)^x} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{(x^{\ln x})'}{\big((\ln x)^x\big)'} = \lim_{x\to\infty}\frac{(x^{\ln x})\left(2\frac{\ln x}{x}\right)}{\big[(\ln x)^x\big]\left(\ln(\ln x) + \frac{1}{\ln x}\right)}.$$

If the original limit is $L \neq 0$, we may divide both sides by L and use the Quotient Law to get

$$1 = \lim_{x\to\infty}\frac{2\frac{\ln x}{x}}{\ln(\ln x) + \frac{1}{\ln x}}. \qquad (*)$$

However we have

$$\lim_{x\to\infty} 2\frac{\ln x}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty}\left(\ln(\ln x) + \frac{1}{\ln x}\right) = \infty,$$

by l'Hospital's Rule and the $\lim_{x\to\infty}\ln x = \infty$, respectively. Therefore the limit on the right-hand side of equation $(*)$ is zero and we get a contradiction, $1 = 0$. Therefore our assumption that the original limit is not zero must be false, that is

$$\lim_{x\to\infty}\frac{x^{\ln x}}{(\ln x)^x} = 0.$$

Remark: Note that we can evaluate the limit without l'Hospital's Rule. Introducing the new variable $u = \ln x$ (and thus $x = e^u$) lets us write

$$\frac{x^{\ln x}}{(\ln x)^x} = \frac{(e^u)^u}{u^{(e^u)}} = \frac{e^{(u^2)}}{\left(e^{\ln u}\right)^{(e^u)}} = \frac{e^{(u^2)}}{e^{(e^u\ln u)}} = e^{(u^2 - e^u\ln u)} \qquad (\dagger)$$

If $x \to \infty$, then $u \to \infty$ and the exponent $u^2 - e^u\ln u$ tends to $-\infty$. We can see this by finding an upper bound for $u^2 - e^u\ln u$ that clearly goes to $-\infty$ as $u \to \infty$. Since $\ln u > 1$ iff $u > e$, we have $u^2 - e^u\ln u < u^2 - e^u$ for all $u > e$. We also have $e^u > u + u^2$ for u large enough. Specifically, the Taylor series expansion for e^u implies that $e^u > \dfrac{u^3}{3!}$ for $u \geq 0$, and thus

$$\frac{u^3}{3!} > u + u^2 \quad\Leftrightarrow\quad u^2 > 6 + 6u \quad\Leftrightarrow\quad (u-3)^2 > 15 \quad\Leftrightarrow\quad u > 3 + \sqrt{15}.$$

So taking $u > 3 + \sqrt{16} = 3 + 4 = 7$ ensures that $e^u > u + u^2$. Thus we have shown that if $u > 7$, then $u^2 - e^u \ln u < u^2 - (u + u^2) = -u$ hence using equation (†) we see that

$$0 \leq \frac{x^{\ln x}}{(\ln x)^x} \leq e^{-u} = \frac{1}{e^u} = \frac{1}{x} \qquad \text{if } x > e^7.$$

(In fact, $x > e^7$ is a very crude estimation as the above inequality already holds when $x > 9$.)

Thus we can conclude that $\lim_{x \to \infty} \frac{x^{\ln x}}{(\ln x)^x} = 0$ by the Squeeze Theorem.

3) Use implicit differentiation to find an equation of the tangent line to the logarithmic spiral $\arctan \frac{y}{x} = \ln \sqrt{x^2 + y^2}$ at the point $\left(\frac{\sqrt{3}}{2} e^{\pi/6}, \frac{1}{2} e^{\pi/6} \right)$.

Solution. Let us write the right-hand side of the equation as $\ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$ and differentiate both sides of the $\arctan \frac{y}{x} = \frac{1}{2} \ln(x^2 + y^2)$ with respect to x. We get

$$\frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{y'x - y}{x^2} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot (2x + 2yy')$$

which is equivalent to

$$\frac{1}{x^2 + y^2} \cdot (y'x - y) = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot (2x + 2yy')$$

which in turn is equivalent to

$$y'x - y = x + yy'.$$

Solving this equation for y' yields

$$y' = \frac{x + y}{x - y} \qquad \text{if } x \neq y.$$

Thus y' at the point $\left(\frac{\sqrt{3}}{2} e^{\pi/6}, \frac{1}{2} e^{\pi/6} \right)$ attains the value $\frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + 2\sqrt{3}$. Therefore an equation for the tangent line is

$$y = (2 + 2\sqrt{3}) \left(x - \frac{\sqrt{3}}{2} e^{\pi/6} \right) + \frac{1}{2} e^{\pi/6}.$$

4) Prove the following statement for every positive integer n.

If $r > 1$, then there exists a $c \in (1, r)$ such that $\frac{1}{n} \sum_{k=0}^{n-1} r^k = c^{n-1}$.

Solution. If $r > 1$, then the n-th partial sum of the geometric series $1 + r + r^2 + \dots$ can be written as

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}$$

for every positive integer n (as seen in the Lectures). The right-hand side as the slope of the secant line for the function $f(x) = x^n$ over the interval $[1, r]$. Being an elementary function f is continuous and differentiable over its domain, which in this case, is the entire number line. Thus f is continuous on $[1, r]$ and differentiable on $(1, r)$. By the Mean Value Theorem there exists a number $c \in (1, r)$ such that

$$f'(c) = \frac{f(r) - f(1)}{r - 1}$$

the left-hand side of which, due to the Power Rule $(x^n)' = nx^{n-1}$, can be written as nc^{n-1}. Therefore we obtain

$$nc^{n-1} = \sum_{k=0}^{n-1} r^k.$$

Dividing both sides by n concludes the proof.

5) Compute the degree 2 Taylor polynomial of $f(x) = \dfrac{x-3}{\sqrt{x^2+3}}$ around $x = -1$.

Solution. The function $f(x)$ is the quotient of two differentiable functions, $F(x) = x - 3$ and $G(x) = \sqrt{x^2+3} = (x^2+3)^{1/2}$ therefore the derivative of $f(x)$ can be calculated using the *Quotient Rule* $[(\frac{F}{G})' = \frac{F'G - FG'}{G^2}]$. We find that

$$\begin{aligned} f'(x) &= \frac{(x-3)'(x^2+3)^{1/2} - (x-3)((x^2+3)^{1/2})'}{((x^2+3)^{1/2})^2} \\ &= \frac{(x^2+3)^{1/2} - (x-3)\frac{1}{2}(x^2+3)^{-1/2}(2x)}{x^2+3} \\ &= \frac{(x^2+3) - x(x-3)}{(x^2+3)^{3/2}} \\ &= \frac{3(x+1)}{(x^2+3)^{3/2}}. \end{aligned} \tag{1}$$

To compute the derivatives in the numerator we used the *Difference Rule* $[(x-3)' = (x)' - (3)']$, the *Power Rule* $[(x^n)' = nx^{n-1}]$, the *Chain Rule* $[((x^2+3)^{1/2})' = \frac{1}{2}(x^2+3)^{-1/2}(x^2+3)']$, and the *Sum Rule* $[(x^2+3)' = (x^2)' + (3)']$.

A similar calculation yields the second derivative

$$\begin{aligned} f''(x) &= \frac{(3x+3)'(x^2+3)^{3/2} - (3x+3)((x^2+3)^{3/2})'}{((x^2+3)^{3/2})^2} \\ &= \frac{3(x^2+3)^{3/2} - (3x+3)\frac{3}{2}(x^2+3)^{1/2}(2x)}{(x^2+3)^3} \\ &= \frac{3(x^2+3) - 3x(3x+3)}{(x^2+3)^{5/2}} \\ &= \frac{-3(2x^2+3x-3)}{(x^2+3)^{5/2}} \end{aligned} \tag{2}$$

Here we applied the same rules of differentiation as before as well as the *Constant Multiple Rule*. An alternative solution involves writing the denominator as $(x^2+3)^{-1/2}$ and applying the *Product Rule* instead of the *Quotient Rule* to find the first and second order derivatives of $f(x)$.

We get $f(-1) = -2$.

Substituting $x = -1$ into (1) yields $f'(-1) = 0$.

Plugging $x = -1$ into (2) yields $f''(-1) = 3/8$.

Therefore the quadratic Taylor polynomial $T_2(x)$ of $f(x)$ around $x = -1$ reads

$$T_2(x) = -2 + \tfrac{3}{16}(x+1)^2 = \tfrac{1}{16}(3x^2 + 6x - 29).$$

6) Find the value $f(2)$ given that the graph of f passes through the point $(1,4)$ and the slope of the tangent line at $\big(x, f(x)\big)$ is $2x + \ln x$.

Solution. Having $f'(x) = 2x + \ln x$ implies that f has the general form

$$f(x) = \int (2x + \ln x)\,dx = x^2 + x\ln x - x + C,$$

where the indefinite integral was found by using the Sum Rule, Power Rule and, in the case of $\int \ln x\,dx$, Integration by Parts (with $u = \ln x$, $dv = dx$). The graph of f passing through the point $(1,4)$ means that $4 = f(1) = 1^2 + 1\ln 1 - 1 + C = C$. Thus the function is

$$f(x) = x^2 + x\ln x - x + 4$$

and therefore

$$f(2) = 2^2 + 2\ln 2 - 2 + 4 = 6 + 2\ln 2.$$